

Ising analogue to compact-star matter

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By constructing an Ising analogue of compact-star matter at sub-saturation density we explored the effect of Coulomb frustration on the nuclear liquid-gas phase transition. Our conclusions is twofold. First, the range of temperatures where inhomogeneous phases form expands with increasing Coulomb-field strength. Second, within the approximation of uniform electron distribution, the limiting point upon which the phase-coexistence region ends does not exhibit any critical behaviour. Possible astrophysics consequences and thermodynamical connections are discussed.

PACS numbers: 26.60.+c, 68.35.Rh, 51.30.+i, 05.50.+q

In absence of Coulombic interactions, the aspect of warm nuclear matter below its saturation density is a dense phase immersed in a low-density gas. The Coulomb repulsion strongly affects this liquid-gas phase transition by forbidding the condensation in macroscopic drops. This phenomenon, originally described in condensed-matter studies with the name of “frustration”, denotes all systems where no states exist where all interaction energies are simultaneously minimised: they range from magnets on specific lattices to liquid crystals, from spin glasses to protein folding.

Nuclear matter in the density range of the liquid-gas phase transition can be produced either in the expansion and disassembling of nuclei involved in violent ion collisions, or in the core of supernovae explosions, as well as in inner neutron-star crusts. It is known since the seventies [1] that the clusterised solid configurations in the inner crust of a neutron star give rise to complex phases [2], which are often quoted as pasta phases because of their suggestive topologies. Recent molecular-dynamics simulations show that these structures may survive also at finite temperature [3]. The fluctuations connected to pasta phases are expected to enhance matter opacity to neutrino scattering with important consequences on the supernova explosion and cooling dynamics [3, 4]. Such a coherent neutrino-matter scattering is not only expected at low temperature, but even more in the possible occurrence of a critical point in the post-bounce supernova explosion, with the associated phenomenon of critical opalescence [4, 5]. At variance with usual-matter properties, the expected increase in the static form factor was not observed in molecular-dynamics simulation of stellar matter at finite temperature [6]. This might be an effect of the Coulomb interaction which also acts in finite nuclei. However, it should be observed that, differently from nuclei, star matter includes also electrons with the role of neutralising the net charge over macroscopic portions.

The motivation of the present work is to provide general insights about the phase-transition phenomenology of neutral systems with Coulombic interactions. We construct an Ising analogue of compact-star matter by adding long-range interactions and a background of electrons to a Lattice-Gas model. Such an approach allows to take into account charge-density fluctuations by exact calculations, and to compare stellar matter with respect to other physical systems subjected to frustration, like hot atomic nuclei[7, 8] and Coulomb-frustrated ferromagnets[9].

Let us consider a neutral system composed of charged particles occupying a cubic lattice of V cells with occupation numbers $n_i = 0, 1$ immersed in a uniform background of negative charge representing the incompressible degenerated gas of electrons [10]. This leads to a charge $q_i = n_i - \bar{n}$ for each site i , where $\bar{n} = \sum_j^V n_j/V$ comes from the uniform background. The schematic Hamiltonian $\mathcal{H}_{N+C} = \mathcal{H}_N + \mathcal{H}_C$ reads

$$\mathcal{H}_N = \frac{\epsilon}{2} \sum_{i,j}^V n_i n_j, \quad \mathcal{H}_C = \frac{\lambda \epsilon}{2} \sum_{i \neq j}^V \frac{q_i q_j}{r_{ij}}, \quad (1)$$

where \sum' extends over closest neighbours. The translational invariance is ensured by imposing the distance \mathbf{r}_{ij} to be the shortest between \mathbf{i} and \mathbf{j} in the periodic space. The short-range (nuclear-like \mathcal{H}_N) and long-range (Coulomb-like \mathcal{H}_C) interactions are characterised by the coupling constants ϵ and $\lambda \epsilon$ respectively, so that λ measures the strength of frustration. To mimic stellar matter the numerical values are set to $\epsilon = -5.5 \text{ MeV}$ and $\lambda \epsilon = \alpha \hbar c \rho_0^{1/3} x^2$, where $\rho_0 = 0.17 \text{ fm}^{-3}$ is the nuclear saturation density, and $x = 1/3$ is a typical proton fraction. \mathcal{H}_C can be rewritten as $\mathcal{H}_C = \frac{\lambda \epsilon}{2} \sum_{i,j}^V n_i n_j C_{ij}$, where $C_{ij} = r_{ij}^{-1} - \bar{r}^{-1}$ if $\mathbf{i} \neq \mathbf{j}$ and $C_{ii} = -\bar{r}^{-1}$ with $\bar{r}^{-1} = \frac{1}{V} \sum_{j \neq 0}^V r_{0j}^{-1}$. Translational invariance imposes $\sum_i^V C_{ij} = 0$. \mathcal{H}_C preserves all the characteristic symmetries of the Ising model and the Coulomb energy is therefore invariant for lattice-occupation reversal. In ref. [9] a complete thermodynamic study of frustration is dedicated to Ising ferromagnets, described by the Hamilto-

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$$\mathcal{H}_{\text{Fe}} = \frac{\epsilon}{2} \sum_{i,j}^V s_i s_j + \frac{\lambda \epsilon}{2} \sum_{i \neq j}^V \frac{s_i s_j}{r_{ij}}, \quad (2)$$

where $s_i = n_i - 1/2$ is a spin variable. The two models are related by:

$$\mathcal{H}_{\text{Fe}} = \mathcal{H}_{\text{N+C}} + \frac{\lambda \epsilon}{2\bar{r}} \mathcal{M}^2 + \mu_c \mathcal{M} - \frac{3\epsilon}{4} V, \quad (3)$$

where $\mathcal{M} = \sum_i^V s_i$ is the magnetisation. We deduce that, contrarily to the standard form of closest-neighbour interactions, the charged Lattice-Gas model, eq.(1), in the grand canonical ensemble cannot be mapped into an Ising-ferromagnet model, eq.(2), in an external magnetic field because of the term in \mathcal{M}^2 in eq.(3). Since this term scales with V^2 , a thermodynamic limit exists for \mathcal{H}_{Fe} only for vanishing magnetisation conversely to $\mathcal{H}_{\text{N+C}}$ [11]. As we will show further on, since \mathcal{M} , or the particle density $\rho = \mathcal{M}/V + 1/2$, is an order parameter of the Ising model, to impose $\mathcal{M} = 0$ as a strict constraint substantially modifies the thermodynamics of the system [12]. Moreover, the presence of the non-linear term \mathcal{M}^2 directly affects the curvature of the order parameter distribution and so modifies the phase properties.

Clarified these distinctions, henceforth we analyse the thermodynamics of the $\mathcal{H}_{\text{N+C}}$ Hamiltonian. To numerically accelerate thermodynamic convergence, the finite lattice is repeated in all three directions of space a large number R of times. Each site \mathbf{i} has then R replicas of itself, each one displaced from \mathbf{i} of a vector $\mathbf{m}L$, where \mathbf{m} has integer components and $L = \sqrt[3]{V}$ is the cubic lattice length. This procedure is equivalent to a renormalisation of the long-range coupling C_{ij} [13].

The multi-canonical ensemble [8] is a specifically adapted statistical tool to deal with frustrated systems. The Hamiltonian components $\mathcal{H}_N, \mathcal{H}_C$ are treated as two independent observables associated to two Lagrange multipliers β_n, β_C , respectively. The multi-canonical partition sum reads

$$Z_{\beta_n, \beta_C}(N, V) = \int W(E_n, E_C, N, V) e^{-\beta_n E_n - \beta_C E_C} dE_n dE_C, \quad (4)$$

where $W(E_n, E_C, N, V)$ is the density of states with nuclear energy E_n , Coulomb energy E_C , and number of particles N . A generalized grand potential is defined by

$$Z_{\beta_n, \beta_C, \alpha}^G(V) = \int Z_{\beta_n, \beta_C}(N, V) e^{\beta_n \mu N} dN. \quad (5)$$

When $\beta_C = \beta_n$ the ensemble coincides with the conventional grandcanonical form. When $\beta_C = 0$ the system reduces to the standard Ising model. Therefore, the multi-(grand)canonical ensemble allows to construct one single

phase diagram both for neutral and charged matter by exploring the space (β_n, β_C) . It should be noticed that varying independently β_n and β_C is equivalent to changing the effective charge $q_{\text{eff}}^2 = \lambda \beta_C / \beta_n$.

The canonical phase diagram in the (β_n, β_C) space can be accessed from the topological properties of the particle density distribution in the corresponding grand-canonical ensemble eq.(5) [14]. For a given value of β_C , Fig. 1A describes, as a function of the inverse temperature β_n , the grand-canonical density distribution calculated at the critical chemical potential $\mu = \mu_c$. This distribution is sampled from eq.(5) with a standard Metropolis technique [14]. At high β_n , the density distribution shows two peaks of the same height, which are associated to two co-existing phases [14]. The solid lines in Fig. 1A mark the low-density and the high-density branches of the distribution ridge associated to the gaseous and liquid phase, respectively. They delimit the liquid-gas coexistence region and join at the limiting temperature above which the distribution stops to be bimodal. The dashed line indicates the coexistence region obtained in the case of an uncharged system $\beta_C = 0$. From a series of similar calculations performed for different values of β_C , we could extract the evolution of the inverse limiting temperature as a function of β_C as plotted in Fig. 1B. We observe an increase of the limiting temperature β_n^{-1} for increasing strength of the Coulomb field.

It is important to notice that many other physical systems subjected to Coulomb frustration exhibit the opposite behaviour. This is notably the case of frustrated Ising ferromagnets, as well as of finite atomic nuclei. In such cases the Coulomb repulsion is known to reduce the limiting temperature [7, 8, 9, 15, 16]. This reduction

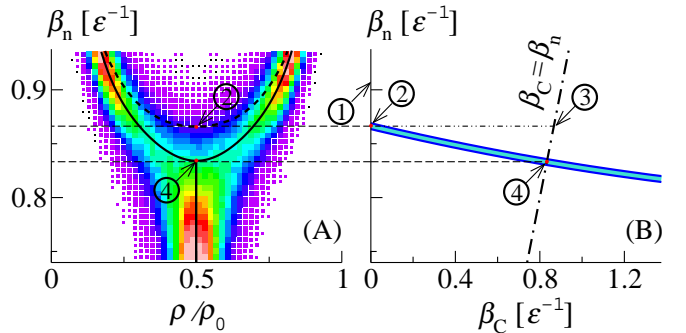


FIG. 1: (Color online) Metropolis calculations ($\mu = \mu_c$, $L = 10$): (A) The logarithmic cluster plot gives the density distributions for different values of β_n and for $\beta_C = 0.83\epsilon$, which is the inverse limiting temperature computed for $\beta_C = \beta_n$. The solid line indicates the ridge of the distribution, the dashed line the coexistence region of an Ising-like system ($\beta_C = 0$). The points (2) and (4) indicate the limiting temperatures for $\beta_C = 0$ and $\beta_C = \beta_n$. (B) Phase diagram in the multicanonical ensemble giving the inverse limiting temperature β_n as a function of β_C . The points (1) and (3) are used in Fig. 2.

is also an usual expectation in the astrophysical context [2, 17, 18, 19]. However, a recent calculation of nuclear-pasta structure within the RMF model [10] indicated a widening of the density range connected to the mixed phase region when the Coulomb field is included under the constraint of charge neutrality over the Wigner-Seitz cells. Our results are consistent with this finding, and suggest that this effect is model independent and should persist at finite temperature[11].

The analysis of the event distribution as a function of their nuclear and Coulomb energy elucidates the physical origin of the increase of the limiting temperature. Let us focus on a few points in the phase diagram of Fig. 1B. Points (1) and (2) describe a path along the line $\beta_C = 0$, where the properties are those of an uncharged system. Indeed, although a non-zero Coulomb energy E_C is calculated, its contribution does not influence the partition probability since $\beta_C = 0$. Clusterised partitions with a high contribution of E_C can then be explored. The corresponding panels (1) and (2) in Fig. 2 show the transition from subcritical to critical partitions.

Introducing $\beta_C \neq 0$ favors small Coulomb energies. Its minimum corresponds to a pure phase since the largely uniform proton-charge distribution is almost exactly compensated by the electron background. Thus pure-phase events dominate the distribution over a temperature range which is expected to increase with β_C .

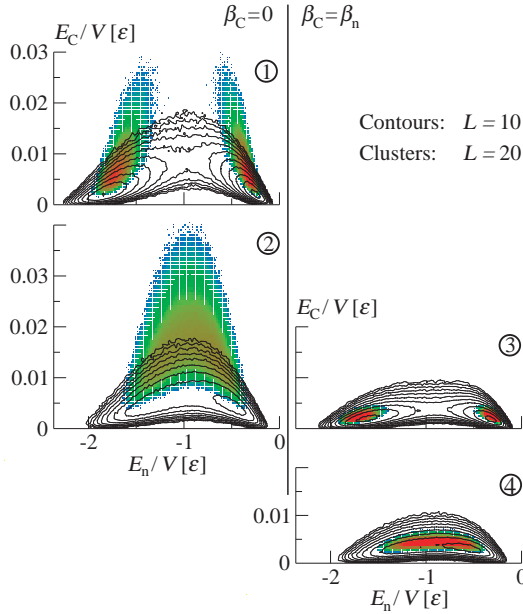


FIG. 2: (Color online) Probability distribution for the two energy components E_n, E_C for the four points indicated in Fig. 1B. Contour plots refer to $L = 10$; logarithmic cluster plots refer to $L = 20$. Points (1) and (3) belong to the coexistence region and manifest bimodal patterns. Both point (2) and (4) are at the limiting temperature but the critical scaling effects exhibited at point (2) are absent at point (4).

For example, when β_C is increased starting from the Ising critical point (point (2) in figures), the mixed partition probability is reduced in favor of pure-phase events. The distribution thus goes back to a bimodal shape and the system is driven back to coexistence. An example is given in panel (3) of Fig. 2 for $\beta_C = \beta_n$. In order to reach again a limiting point where the bimodality disappears, the temperature β_n^{-1} should be further increased as illustrated at point (4) taken on the line $\beta_C = \beta_n$.

The question now arises as to whether the limiting points of the frustrated system are second-order critical points like in the uncharged (or Ising) system. Criticality arises from the divergence of the correlation length ξ . It describes the exponential decay of the correlation function $\sigma(\mathbf{r}_{ij}) = \langle \delta n_i \delta n_j \rangle = \langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle$ according to the expression $\sigma(r) \propto e^{-r/\xi} \cdot r^{-(D-2+\eta)}$, where D is the space dimension and η is a critical exponent. Using the properties of the C_{ij} matrix, $\sigma(r)$ can be easily related to the mean Coulomb-energy density by

$$\left\langle \frac{E_C}{V} \right\rangle = \frac{\sigma(0)}{\bar{r}} + \frac{\lambda \epsilon}{2} \sum_{j \neq 0}^V \frac{\sigma(\mathbf{r}_{0j})}{r_{0j}}. \quad (6)$$

Eq.(6) indicates that a diverging correlation length at the critical point manifests by a divergent Coulomb-energy density. When $\beta_C \neq 0$ the only solution for the system to avoid such a singularity is to suppress the second-order critical character of the limiting point [11]. In this case, the correlation length keeps a finite value at the limiting temperature and the thermodynamic limit is fulfilled by a constant value of $\langle E_C/V \rangle$. Such a behaviour is tested in Fig. 2, where the same calculation is repeated for different lattice sizes: the average Coulomb energy is seen to increase with the lattice size in the Ising system. This signs a diverging correlation length at the limiting temperature, while Coulomb-energy fluctuations appear quenched in the frustrated system.

These assertions can be formally verified in terms of critical exponents. If the asymptotic value of the limiting temperature $T_{\text{lim}} = \lim_{L \rightarrow \infty} \dot{T}_{\text{lim}}(L)$ corresponds to a critical point, finite-size scaling insures that $\dot{T}_{\text{lim}}(L)$ should evolve as $\dot{T}_{\text{lim}}(L) - T_{\text{lim}} \propto L^{-1/\nu}$ [20]. Fig. 3A and 3B illustrate that, while the scaling is respected with the Ising value of ν when $\beta_C = 0$, ν should be infinitely large to describe the frustrated system. Since ν rules the divergence of the correlation length at the critical point, $\xi \propto t^{-\nu}$ with $t = T/T_{\text{lim}} - 1$, this is a first indication of a finite correlation length for the charged system. We then test the finite-size scaling theory [20] on the quantity $\chi = \sum_{ij}^N \sigma_{ij}/T$. This form of χ represents the susceptibility only for $T \geq \dot{T}_{\text{lim}}(L)$. In this case, it should scale with the critical exponent γ as $L^{-\gamma/\nu} \chi = f(L^{-1/\nu} t)$ if the ending point of the coexistence zone represented a critical point; the function $f(L^{-1/\nu} t)$ should be constant when $\xi \sim L$ and $f(L^{-1/\nu} t) \propto (L^{1/\nu} t)^{-\gamma}$ when $\xi \ll L$. When $T < \dot{T}_{\text{lim}}(L)$, χ contains also jumps between the

low-density, $\langle n \rangle_G$ (gas), and the high-density, $\langle n \rangle_L$ (liquid), solutions and should therefore scale with the critical exponent β defined from the scaling of the order parameter $\langle n \rangle_L - \langle n \rangle_G \propto t^\beta$ as $\chi \propto L^D t^{2\beta}$. By introducing the hyperscaling relation $D = (\gamma + 2\beta)/\nu$, all scaling laws condense in the form

$$L^{-\gamma/\nu} \chi \propto \begin{cases} \text{constant} & \text{as } \xi \sim L \\ (L^{1/\nu} t)^{-\gamma} & \text{as } \xi \ll L, T \geq \hat{T}_{\text{lim}}(L) \\ (L^{1/\nu} t)^{2\beta} & \text{as } \xi \ll L, T < \hat{T}_{\text{lim}}(L) \end{cases} \quad (7)$$

Fig. 3C illustrates the perfect consistency of the $\beta_C = 0$ system with Ising critical exponents. Inversely, for the charged system, no combination of β , γ and ν can be found to let all calculated point collapse on the same law. In particular, for $\gamma/\nu = 0$ and $\nu \rightarrow \infty$, only the points for $T \gg \hat{T}_{\text{lim}}(L)$ collapse on one line, while those for $T \ll \hat{T}_{\text{lim}}(L)$ disperse as shown in Fig. 3D. This demonstrates that the effect of the Coulomb field is not a simple increase of the ν exponent, but a complete quenching of criticality for the frustrated system. The loss of critical behaviour has been already observed in Ising models for the Ising frustrated ferromagnet, eq.(2), where the coexistence region was seen to end at a first-order point [9]. This effect was not reproduced by the mean-field approximation and was attributed to fluctuations.

Our conclusion is twofold. First, we observed that, at variance with most predictions such as mean-field ex-

pectations [17, 18, 19], in presence of a uniform electron background, the Coulomb field originating from charge-density fluctuations results in increasing the limiting temperature for phase coexistence. Therefore, the mixed-phase phenomenology may be relevant for the proto-neutron-star structure in a wider temperature-range than usually expected [2]. Second, the Coulomb field suppresses the critical character of the limiting temperature. For this reason we expect warm stellar matter to show relatively small opacity to neutrino scattering in agreement with the findings of ref. [21], which may have important consequences on the cooling dynamics [4].

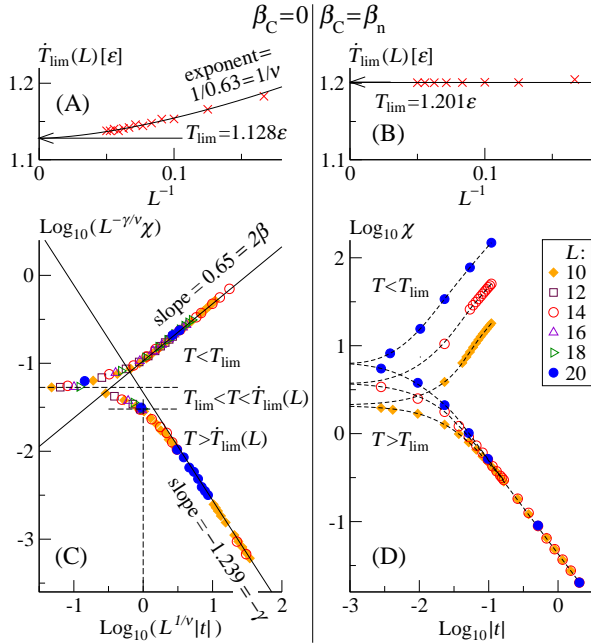


FIG. 3: (Color online). Panels A,B. Study of the rounding as a function of the linear size L , and extrapolation towards the thermodynamic limit. Panels C,D. Study of the finite-size scaling eq.(7). All calculations are presented for the uncharged ($\beta_C = 0$) and the charged system ($\beta_C = \beta_n$) with proton fraction $x = 1/3$.

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